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Minimal energy of unicyclic graphs of a given diameter

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The energy of a graph is defined as the sum of the absolute values of all the eigenvalues of the graph. For a given positive integer d with $3 \leqslant d \leqslant n-2$, we characterize the graphs with minimal energy in the class of unicyclic graphs with n vertices and a given diameter d.

KEY WORDS: energy, diameter, unicyclic graphs, characteristic polynomial

AMS subject classification: 05C50, 05C35

1. Introduction

Let G be a simple graph with n vertices. The characteristic polynomial of G, denoted by $\phi(G)$, is defined as

$$\phi(G) = \det(xI - A(G)) = \sum_{i=0}^{n} a_i(G)x^{n-i},$$

where I is the identity matrix of order n and $A(G)$ is an adjacency matrix of G. The roots of the equation $\phi(G) = 0$, denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$, are the eigenvalues of the graph G. Since $A(G)$ is symmetric, all eigenvalues of G are real. The energy of G, denoted by $E(G)$, is then defined by

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

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In chemistry, the energy of a given molecular graph is of interest since it is closely related to the total π-electron energy of the molecule represented by that graph. See Refs. [1–3] for more details on graph-energy concept and a survey of the mathematical properties and results.

For a graph G with n vertices, let $b_i(G) = |a_i(G)|$, $i = 0, 1, \ldots, n$. Note that $b_0(G) = 1$, $b_1(G) = 0$, and $b_2(G)$ is the number of edges of G. For convenience, let $b_i(G) = 0$ if $i < 0$. Let $m(G, k)$ be number of k-matchings of G. If G is an acyclic graph, then [1] $b_{2k}(G) = m(G, k)$ and $b_{2k+1}(G) = 0$ for all k.

A connected graph with n vertices and n edges is called a unicyclic graph. Obviously, a unicyclic graph has exactly one cycle.

Let $\mathcal{G}(n)$ be the class of graphs with n vertices whose components are all trees except at most one being a unicyclic graph. That is, any graph in $\mathcal{G}(n)$ is either acyclic or contains exactly one cycle. By Sachs theorem and the Coulson integral formula [1, 2], we have [4, 5]

$$E(G) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}x}{x^2}\ln\left[\left(\sum_{j=0}^{\lfloor n/2\rfloor} b_{2j}(G)x^{2j}\right)^2 + \left(\sum_{j=0}^{\lfloor n/2\rfloor} b_{2j+1}(G)x^{2j+1}\right)^2\right]. \quad (1)$$

Thus $E(G)$ is a monotonically increasing function of $b_i(G)$, $i = 1, 2, \ldots, n$. Let $G_1, G_2 \in \mathcal{G}(n)$. If $b_i(G_1) \geqslant b_i(G_2)$ for all $i \geqslant 0$, then we write $G_1 \succeq G_2$. If $G_1 \succeq G_2$ and there is an i_0 such that $b_{i_0}(G_1) > b_{i_0}(G_2)$, then we write $G_1 \succ G_2$. So from (1) we have the following increasing property of energy:

$$G_1 \succ G_2 \Rightarrow E(G_1) > E(G_2). \quad (2)$$

This increasing property of energy has been used in the study of extremal values of energy over some classes of graphs. For instance, Gutman [4] determined the trees with minimal and maximal energies. Hou [5] determined the unicyclic graphs with minimal energy, and Yan and Ye [6] determined trees of a given diameter with minimal energy. More results in this direction can be found in Refs. [7–12].

Let $\mathcal{U}(n, d)$ be the class of unicyclic graphs with n vertices and diameter d, where $2 \leqslant d \leqslant n-2$. By the result of Ref. [5], the graph obtained by attaching $n-3$ pendant edges to a vertex of a triangle is the unique graph in $\mathcal{U}(n, 2)$ with minimal energy. In this paper, we will prove that for $3 \leqslant d \leqslant n-2$, the graph $U_{n,d}$ is the unique graph in $\mathcal{U}(n, d)$ with minimal energy, where the graph $U_{n,d}$ is shown in figure 1.

2. Preliminaries

Denote by P_n the path with n vertices. For two graphs G and H, $G \neq H$ means G and H are not isomorphic, and $G \supseteq H$ means G contains H as a subgraph.

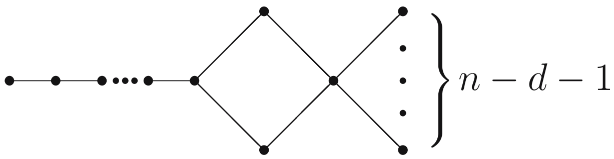

Figure 1. Graph $U_{n,d}$ with $d \geqslant 3$.

Lemma 1. Let G be a graph in $\mathcal{G}(n)$.

(a) If G contains exactly one cycle C_r and uv is an edge on this cycle, then

$$b_i(G) = b_i(G - uv) + b_{i-2}(G - u - v) - 2b_{i-r}(G - C_r) \text{ if } r \equiv 0 \pmod 4,$$
$$b_i(G) = b_i(G - uv) + b_{i-2}(G - u - v) + 2b_{i-r}(G - C_r) \text{ if } r \not\equiv 0 \pmod 4.$$

(b) If uv is a cut edge of G, then

$$b_i(G) = b_i(G - uv) + b_{i-2}(G - u - v).$$

Proof. For edge uv of G, it is known [1, 13] that

$$\phi(G) = \phi(G - uv) - \phi(G - u - v) - 2\sum_{C\in\mathcal{C}} \phi(G - C),$$

where $\mathcal{C}$ is the set of cycles of G containing uv. In particular, if uv is a cut edge, then

$$\phi(G) = \phi(G - uv) - \phi(G - u - v).$$

Now (a) and (b) follow by equating coefficients of x^{n-2k} on both sides of identities above. □

Remark. If uv is a pendant edge with pendant vertex u, then lemma 1(b) becomes

$$b_i(G) = b_i(G - u) + b_{i-2}(G - u - v).$$

Lemma 2. Let G be an acyclic graph and G' a spanning subgraph (resp. proper spanning subgraph) of G. Then $G \succeq G'$ (resp. $\succ G'$).

Lemma 3. Let G be a unicyclic graph and G' a graph obtained from G by deleting at least one edge outside its unique cycle. Then $G \succ G'$.

Proof. Let $H \in \{G, G'\}$ and uv be an edge on the unique cycle C_r in H.

If $r \not\equiv 0 \pmod 4$, by lemma 1 (a)

$$b_i(H) = b_i(H - uv) + b_{i-2}(H - u - v) + 2b_{i-r}(H - C_r).$$

Since G' is a proper spanning subgraph of G, we have by lemma 2 that $b_i(G-uv) \geqslant b_i(G'-uv)$, $b_{i-2}(G-u-v) \geqslant b_{i-2}(G'-u-v)$, and $b_{i-r}(G-C_r) \geqslant b_{i-r}(G'-C_r)$. So we have $b_i(G) \geqslant b_i(G')$, and $b_2(G) > b_2(G')$, and then $G \succ G'$.

If $r \equiv 0 \pmod 4$, then $b_{2k+1}(H) = 0$. By lemma 1 (a)

$$\begin{aligned} b_{2k}(H) =&\ b_{2k}(H-uv) + b_{2k-2}(H-u-v) - 2b_{2k-r}(H-C_r) \\ =&\ m(H,k) - 2m(H-C_r, k-\frac{r}{2}), \end{aligned}$$

i.e., $b_{2k}(H)$ is the number of k-matchings of H that contain at most $r/2-1$ edges of the cycle C_r. So $b_{2k}(H) = \sum_S m(H_S, k-|S|)$, where summation goes over all non-perfect matchings S of C_r, and where H_S is the graph obtained from H by deleting the end vertices of S and other edges of C_r. Since for any S, G'_S is a subgraph of G_S, we have $m(G_S, k-|S|) \geqslant m(G'_S, k-|S|)$ and then $b_{2k}(G) \geqslant b_{2k}(G')$ for all k. Note that $b_2(G) > b_2(G')$. We have $G \succ G'$. □

Lemma 4. [1, 12]. For $2 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$ and $n \geqslant 4$,

$$P_n \succ P_i \cup P_{n-i} \succ P_1 \cup P_{n-1}.$$

Let $\mathcal{T}(n,d)$ be the class of trees with n vertices and diameter d, where $2 \leqslant d \leqslant n-2$. Let $B_{n,d}$ be a graph obtained from the path P_d by attaching $n-d$ pendant edges to an end vertex of P_d. Let $Y_n = B_{n,3}$. Let $K_{1,n-1}$ be the star with n vertices.

Lemma 5. [4]. For $n \geqslant 5$,

$$P_n \succ Y_n \succ K_{1,n-1}.$$

Lemma 6. [6]. Let $T \in \mathcal{T}(n,d)$ and $T \neq B_{n,d}$. Then $T \succ B_{n,d}$.

Lemma 7. If $d > d_0 \geqslant 3$, then $B_{n,d} \succ B_{n,d_0}$.

Proof. Note that for any integer $k \geqslant 1$,

$$\begin{aligned} m(B_{n,d},k) =&\ m(B_{n-1,d-1},k) + m(B_{n-2,d-2},k-1), \\ m(B_{n,d-1},k) =&\ m(B_{n-1,d-1},k) + m(P_{d-2},k-1). \end{aligned}$$

Since P_{d-2} is a proper subgraph of $B_{n-2,d-2}$, we have $m(B_{n-2,d-2},k-1) \geqslant m(P_{d-2},k-1)$ and this inequality strict for $k=2$. Hence $m(B_{n,d},k) \geqslant m(B_{n,d-1},k)$ and this inequality strict for $k=2$. It follows that $B_{n,d} \succ B_{n,d-1} \succ \cdots \succ B_{n,d_0}$. □

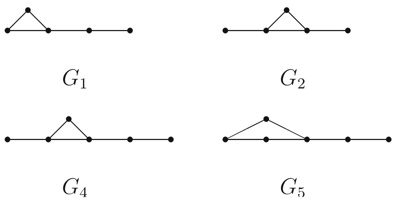

Figure 2. Graphs G_i, $i = 1, \ldots, 5$.

For minimal energies in $\mathcal{T}(n, d)$, we point out that in [12, theorem 1] a condition $T \neq T(n, d; 1, 0, \ldots, 0, n-d-2)$ should be added.

By lemma 1 (b), it is easy to see that the following lemma holds. It is known [9] that similar result holds for bipartite graphs.

Lemma 8. Let $G, G' \in \mathcal{G}(n)$. Let uv (resp. $u'v'$) be a pendant edge with the pendant vertex u (resp. u') of the graph G (resp. G'). If $G-u \succeq G'-u'$, and $G-u-v \succ G'-u'-v'$, or $G-u \succ G'-u'$, and $G-u-v \succeq G'-u'-v'$, then $G \succ G'$.

3. Main results

Now we consider the minimal energy of graphs in $\mathcal{U}(n, d)$ with $3 \leqslant d \leqslant n-2$. We first consider the case $d = n-2$.

Lemma 9. Let $G \in \mathcal{U}(n, n-2)$ with $n \geqslant 5$ and $G \neq U_{n,n-2}$. Then $G \succ U_{n,n-2}$.

Proof. We prove the lemma by induction on n.
If $n = 5$, then G is isomorphic to G_1 or G_2. (See figure 2.) It is easy to see that

$$\phi(G_1) = x^5 - 5x^3 - 2x^2 + 4x + 2, \qquad \phi(G_2) = x^5 - 5x^3 - 2x^2 + 3x,$$
$$\phi(U_{5,3}) = x^5 - 5x^3 + 2x.$$

It is obvious that $G_i \succ U_{5,3}$ for $i = 1, 2$.

If $n = 6$, then G is isomorphic to G_3, G_4 or G_5. (See figure 2.) Note that

$$\phi(G_3) = x^6 - 6x^4 - 2x^3 + 8x^2 + 4x - 1, \qquad \phi(G_5) = x^6 - 6x^4 + 6x^2,$$
$$\phi(G_4) = x^6 - 6x^4 - 2x^3 + 7x^2 + 2x - 1, \qquad \phi(U_{6,4}) = x^6 - 6x^4 + 5x^2.$$

It is obvious that $G_i \succ U_{6,4}$ for $i = 3, 4, 5$.

Suppose that the result holds for graphs in $\mathcal{U}(n-1, n-3)$ and $\mathcal{U}(n-2, n-4)$ and that $G \in \mathcal{U}(n, n-2)$ and $G \neq U_{n,n-2}$ where $n \geqslant 7$.

Let u (resp. u') be a pendant vertex, adjacent to v (resp. v'), which has the largest distance to a vertex on the unique cycle of G (resp. $U_{n,n-2}$). Then the degree of v is 2. So $G - u \in \mathcal{U}(n-1, n-3)$, $G - u - v \in \mathcal{U}(n-2, n-4)$, and $U_{n,n-2} - u' = U_{n-1,n-3}$, $U_{n,n-2} - u' - v' = U_{n-2,n-4}$.

Since $G \neq U_{n,n-2}$, we have $G-u \neq U_{n-1,n-3}$ or $G-u-v \neq U_{n-2,n-4}$. By the induction assumption, we have either $G-u \succ U_{n-1,n-3}$ and $G-u-v \succeq U_{n-2,n-4}$, or $G-u \succeq U_{n-1,n-3}$ and $G-u-v \succ U_{n-2,n-4}$. By lemma 8, $G \succ U_{n,n-2}$. □

Now we are ready to prove our main result:

Theorem 1. Let $G \in \mathcal{U}(n,d)$ with $d \geqslant 3$ and $G \neq U_{n,d}$. Then $E(G) > E(U_{n,d})$.

Proof. By the increasing property (2) of energy, it suffices to prove that $G \succ U_{n,d}$. We prove this by induction on $n-d$.

By lemma 9, the result holds for $n-d=2$.

Let $p \geqslant 3$ and suppose that the result holds for $n-d < p$. Now suppose that $n-d=p$. Let u' be the vertex of degree 3 in $U_{n,d}$ and v' a vertex on the quadrangle that is adjacent to u'. By lemma 1 (a)

$$b_i(U_{n,d}) = b_i(B_{n,d}) + b_{i-2}(P_{d-3} \cup K_{1,n-d}) - 2b_{i-4}(P_{d-3}).$$

First note that $b_n(C_n) \geqslant 0 = b_n(U_{n,d})$, and that if $0 \leqslant i \leqslant n-1$, then by lemma 1 (a)

$$b_i(C_n) = b_i(P_n) + b_{i-2}(P_{n-2}).$$

By lemma 7, $P_n \succ B_{n,d}$ and by lemma 4, $P_{n-2} \succeq P_{d-3} \cup P_{n-d+1} \succ P_{d-3} \cup K_{1,n-d}$. So $b_i(C_n) \geqslant b_i(U_{n,d})$ and $b_4(C_n) > b_4(U_{n,d})$. Thus $C_n \succ U_{n,d}$.

In the following, suppose that the unique cycle of G is C_r where $r < n$. Let $P(G) = v_0, v_1, \ldots, v_d$ be a diametrical path of G. Then one of v_0, v_d must be a pendant vertex.

Suppose that all pendant vertices are on $P(G)$. Since $p \geqslant 3$, there are at least two adjacency vertices u and v on C_r which are outside $P(G)$ such that $G-uv \in \mathcal{T}(n,d_1)$, $G-v \in \mathcal{T}(n-1,d_2)$ and $G-u-v \in \mathcal{T}(n-2,d_3)$, where $d_1, d_2, d_3 \geqslant d$. By lemmas 6 and 7, $G-uv \succeq B_{n,d_1} \succeq B_{n,d}$, $G-v \succeq B_{n-1,d_2} \succeq B_{n-1,d}$, and $G-u-v \succeq B_{n-2,d_3} \succeq B_{n-2,d}$.

If $r \not\equiv 0 \pmod 4$, then

$$b_i(G) = b_i(G-uv) + b_{i-2}(G-u-v) + 2b_{i-r}(G-C_r).$$

By lemmas 2 and 5, $G-u-v \succeq B_{n-2,d} \succ P_{d-3} \cup Y_{n-d-1} \succ P_{d-3} \cup K_{1,n-d}$. Then $b_i(G) \geqslant b_i(U_{n,d})$ and $b_r(G) > b_r(U_{n,d})$. Thus $G \succ U_{n,d}$.

If $r \equiv 0 \pmod 4$, then by lemma 1 (a), the expression for $b_i(U_{n,d})$ above may be written as

$$b_i(U_{n,d}) = b_i(B_{n-1,d}) + b_{i-2}(P_{d-2}) + b_{i-2}(P_{d-3} \cup K_{1,n-d-1}).$$

Suppose that $r \neq 4$. Then u and v can be chosen such that there is a vertex w on C_r but outside $P(G)$, v is adjacent to u and w, and $G-u-v-w \in$

$\mathcal{T}(n-3, d_4)$ with $d_4 \geqslant d$. By lemmas 6 and 7, $G-u-v-w \succeq B_{n-3,d_4} \succeq B_{n-3,d}$, Let u_1 (resp. w_1) be the vertex adjacent to u (resp. w) in C_r and $u_1 \neq v$ (resp. $w_1 \neq v$). By lemma 1 (b), $b_{i-4}(G-u_1-u-v-w) \geqslant b_{i-r}(G-C_r)$, $b_{i-4}(G-u-v-w-w_1) \geqslant b_{i-r}(G-C_r)$, and then

$$\begin{aligned} b_i(G-uv) &= b_i(G-v)+b_{i-2}(G-v-w)\\ &= b_i(G-v)+b_{i-2}(G-u-v-w)\\ &\quad +b_{i-4}(G-u_1-u-v-w)\\ &\geqslant b_i(G-v)+b_{i-2}(G-u-v-w)+b_{i-r}(G-C_r),\\ b_{i-2}(G-u-v) &= b_{i-2}(G-u-v-w)+b_{i-4}(G-u-v-w-w_1)\\ &\geqslant b_{i-2}(G-u-v-w)+b_{i-r}(G-C_r). \end{aligned}$$

So by lemma 1 (a)

$$\begin{aligned} b_i(G) &= b_i(G-uv)+b_{i-2}(G-u-v)-2b_{i-r}(G-C_r)\\ &\geqslant b_i(G-v)+2b_{i-2}(G-u-v-w)\\ &\geqslant b_i(B_{n-1,d})+2b_{i-2}(B_{n-3,d}). \end{aligned}$$

By lemmas 2 and 5, $B_{n-3,d} \succeq P_{d-3} \cup Y_{n-d} \succ P_{d-3} \cup K_{1,n-d-1}$. Then $b_{i-2}(B_{n-3,d}) \geqslant b_{i-2}(P_{d-3} \cup K_{1,n-d-1})$. Note that $b_{i-2}(B_{n-3,d}) \geqslant b_{i-2}(P_{d-2})$ and $b_2(B_{n-3,d}) > b_2(P_{d-2})$. So $b_i(G) \geqslant b_i(U_{n,d})$ and $b_4(G) > b_4(U_{n,d})$. Thus $G \succ U_{n,d}$.

Suppose that $r = 4$. If all edges on the cycle are not on $P(G)$, then by a similar reasoning as above, we have $G \succ U_{n,d}$. Otherwise, there is exactly one edge on both C_4 and $P(G)$. Then $n = d+3$, and by lemmas 1 and 4, for some j, $1 \leqslant j \leqslant d-1$, we have

$$\begin{aligned} b_i(G) &= b_i(G-uv)+b_{i-2}(G-u-v)-2b_{i-4}(G-C_4)\\ &= b_i(G-v)+b_{i-2}(P_j \cup P_{d+1-j})+b_{i-2}(P_{d+1})-2b_{i-4}(P_j \cup P_{d-1-j})\\ &= b_i(G-v)+b_{i-2}(P_j \cup P_{d-j})+b_{i-4}(P_j \cup P_{d-j-1})\\ &\quad +b_{i-2}(P_d)+b_{i-4}(P_{d-1})-2b_{i-4}(P_j \cup P_{d-1-j})\\ &\geqslant b_i(G-v)+b_{i-2}(P_j \cup P_{d-j})+b_{i-2}(P_d)\\ &\geqslant b_i(B_{n-1,d})+b_{i-2}(P_j \cup P_{d-j})+b_{i-2}(P_d). \end{aligned}$$

By lemma 4, $P_j \cup P_{d-j} \succeq P_1 \cup P_{d-1} \succ 2P_1 \cup P_{d-2}$ and $P_d \succeq P_{d-3} \cup P_3$. Then $b_i(G) \geqslant b_i(U_{n,d})$. Since $b_2(P_d) > b_2(P_{d-3} \cup P_3)$, we have $b_4(G) > b_4(U_{n,d})$. Thus $G \succ U_{n,d}$.

Now suppose that there is at least one pendant vertex outside $P(G)$. Let u' be a pendant vertex of $U_{n,d}$ adjacent to the vertex v' of degree $n-d+1$. Then $U_{n,d}-u' = U_{n-1,d}$ and $U_{n,d}-u'-v' = (n-d-2)P_1 \cup B_{d,d-2}$.

Case 1. There is a pendant vertex u outside $P(G)$ such that its neighbor v lies on C_r. Then $G-u \in \mathcal{U}(n-1,d)$.

If v lies outside $P(G)$, then $G-u-v \supseteq P_{d+1}$. So $G-u-v \succeq (n-d-3)P_1 \cup P_{d+1} \succ (n-d-2)P_1 \cup B_{d,d-2}$.

Suppose that v is on $P(G)$. Then $P(G)$ and C_r have common vertices, say $v_l, \ldots, v_{l+s}$ with $s \geqslant 0$.

If $s = 0$, i.e., $v = v_l$ is the unique common vertex of $P(G)$ and the cycle C_r, then $G - u - v \supseteq P_l \cup P_{d-l} \cup P_2$, and since

$$\begin{aligned} b_i(P_l \cup P_{d-l} \cup P_2) &= b_i(P_l \cup P_{d-l}) + b_{i-2}(P_l \cup P_{d-l}) \\ &\geqslant b_i(P_{d-1}) + b_{i-2}(P_{d-3}) = b_i(B_{d,d-2}) \end{aligned}$$

and $b_4(P_l \cup P_{d-l} \cup P_2) > b_4(B_{d,d-2})$, we have $G - u - v \succeq (n-d-4)P_1 \cup P_l \cup P_{d-l} \cup P_2 \succ (n-d-2)P_1 \cup B_{d,d-2}$.

Suppose that $s > 0$. If $v \neq v_l, v_{l+s}$, then $G - u - v \supseteq P_{d+1}$. So $G - u - v \succeq P_{d+1} \succ P_1 \cup B_{d,d-2}$. Otherwise, for $j = l$ or $l + s$, say $j = l$, $G - u - v \supseteq P_j \cup T_1$, where T_1 obtained by attaching a pendant edge to vertex v_{j+s} of the path $v_{j+1}, \ldots, v_d$. Note that

$$\begin{aligned} b_i(P_j \cup T_1) &= b_i(P_j \cup P_{d-j}) + b_{i-2}(P_j \cup P_{s-1} \cup P_{d-j-s}) \\ &\geqslant b_i(P_{d-1}) + b_{i-2}(P_{d-3}) = b_i(B_{d,d-2}). \end{aligned}$$

If $(j, s) \neq (1, 2)$, then $b_6(P_j \cup T_1) > b_6(B_{d,d-2})$, otherwise $P_j \cup T_1$ is a proper subgraph of $G - u - v$. Hence we have $G - u - v \succ (n-d-2)P_1 \cup B_{d,d-2}$.

Now we have proved that $G - u - v \succ U_{n,d} - u' - v'$. By the induction hypothesis, $G - u \succeq U_{n-1,d}$. By lemma 8, $G \succ U_{n,d}$.

Case 2. The neighbor of any pendant vertex outside $P(G)$ also lies outside C_r. If there is a pendant vertex u, adjacent to a vertex v outside $P(G)$. Then $G-u-v \supseteq C_r \cup P_{d+1}$ or $G-u-v \supseteq G'$ where $G' \in \mathcal{U}(s, d)$, $d+2 \leqslant s \leqslant n-2$. If every pendant vertex outside $P(G)$ is adjacent a vertex on $P(G)$, then we choose a pendant vertex u, adjacent to $v = v_j$, such that $G - u - v \supseteq C_r \cup P_j \cup P_{d-j}$ or $G - u - v \supseteq P_j \cup G''$, where $G'' \in \mathcal{U}(s, d')$ for some s and d' with $s + j \leqslant n - 2$ and $d' \geqslant d - j - 1$. Hence there are three possibilities: $G-u-v \supseteq C_r \cup P_j \cup P_{d-j}$, $G - u - v \supseteq G'$ or $G - u - v \supseteq P_j \cup G''$.

First suppose that $G - u - v \supseteq C_r \cup P_j \cup P_{d-j}$. Then by lemma 1 (a)

$$\begin{aligned} b_i(C_r \cup P_j \cup P_{d-j}) &\geqslant b_i(P_r \cup P_j \cup P_{d-j}) + b_{i-2}(P_{r-2} \cup P_j \cup P_{d-j}) \\ &\quad -2b_{i-r}(P_j \cup P_{d-j}) \\ &\geqslant b_i(P_{r-1} \cup P_j \cup P_{d-j}) + b_{i-2}(P_{r-3} \cup P_j \cup P_{d-j}) \\ &\geqslant b_i(P_{r-1} \cup P_{d-1}) \geqslant b_i(P_{d+r-3}) \geqslant b_i(B_{d,d-2}). \end{aligned}$$

By lemma 3, $G-u-v \succeq (n-d-r-2)P_1 \cup C_r \cup P_j \cup P_{d-j} \succ (n-d-2)P_1 \cup B_{d,d-2}$.

Now suppose that $G-u-v \supseteq G'$. By lemma 3 and the induction hypothesis, $G - u - v \succeq (n-s-2)P_1 \cup U_{s,d} \succeq (n-d-2)P_1 \cup U_{d+2,d}$. By lemma 1

$$\begin{aligned} b_i(U_{d+2,d}) &= b_i(B_{d+2,d}) + b_{i-2}(P_{d-3} \cup P_3) - 2b_{i-4}(P_{d-3}) \\ &= b_i(B_{d+2,d}) + b_{i-2}(P_{d-3}) \geqslant b_i(B_{d,d-2}). \end{aligned}$$

where $b_2(U_{d+2,d}) > b_2(B_{d,d-2})$. So $G-u-v \succeq (n-d-2)P_1 \cup U_{d+2,d} \succ (n-d-2)P_1 \cup B_{d,d-2}$.

Finally suppose that $G - u - v \supseteq P_j \cup G''$. By lemma 3 and the induction hypothesis, $G - u - v \succeq (n-3-d+j)P_1 \cup U_{d-j+1,d-j-1}$. Note that

$$\begin{aligned} b_i(P_j \cup U_{d-j+1,d-j-1}) &= b_i(P_j \cup B_{d-j+1,d-j-1}) + b_{i-2}(P_j \cup P_3 \cup P_{d-j-4}) \\ &\quad -2b_{i-4}(P_j \cup P_{d-j-4}) \\ &= b_i(P_j \cup P_{d-j}) + b_{i-2}(P_j \cup P_{d-j-2}) \\ &\quad +b_{i-2}(P_j \cup P_{d-j-4}) \\ &\geqslant b_i(P_{d-1}) + b_{i-2}(P_{d-3}) = b_i(B_{d,d-2}) \end{aligned}$$

and $b_2(P_j \cup U_{d-j+1,d-j-1}) > b_2(B_{d,d-2})$. We have $G-u-v \succ (n-d-2)P_1 \cup B_{d,d-2}$.

Now we have proved that $G - u - v \succ U_{n,d} - u' - v'$. By the induction hypothesis, $G - u \succeq U_{n-1,d}$. By lemma 8, $G \succ U_{n,d}$.

Combining Cases 1 and 2, we conclude that the result holds for $G \in \mathcal{U}(n,d)$ for $d \geqslant 3$. The theorem is thus proved. □

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